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Causality and indefiniteness of charge in spin $\frac{3}{2}$ field theories

J Prabhakaran, M Seetharaman and P M Mathews
Department of Theoretical Physics, University of Madras, Madras 600025, India

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Abstract. The possibility of retaining causality of propagation of classical spin $\frac{3}{2}$ fields in the presence of an external electromagnetic interaction is investigated. It is shown that in the case of the mixed spin $\frac{3}{2}$–spin $\frac{1}{2}$ theory of Bhabha and Gupta, causality in the interacting situation may be retained by giving up the positive definiteness of the *free* total charge and making an appropriate choice of the arbitrary parameters appearing in the Bhabha–Gupta Lagrangian. We demonstrate that the causal spin $\frac{3}{2}$ equation proposed recently by Fisk and Tait also shares the same feature, namely the indefiniteness of the free total charge.

1. Introduction

The description of spin $\frac{3}{2}$ particles in interaction with prescribed external fields is beset with difficulties. The classic work of Johnson and Sudarshan (1961) showed for the first time that the anticommutators of q-number spin $\frac{3}{2}$ fields minimally coupled to an external electromagnetic field will not be positive in all Lorentz frames. More recently Velo and Zwanziger (1969a) and others (Shamaly and Capri 1972, Madore and Tait 1973, Singh 1973, Tait 1973, Mathews *et al* 1974) have drawn attention to the fact that troubles arise even at the c-number level. In particular, it has been shown that the wavefronts of the classical solutions of the wave equations propagate with speeds greater than that of light even in weak external fields.

The wave equations considered in the above-mentioned works have one feature in common: the total charge of *free* spin $\frac{3}{2}$ fields is positive definite in the space of the solutions of the wave equation. In the course of a study of alternative formulations of spin $\frac{3}{2}$ theories, the authors found that it is possible to avoid acausality of propagation in the case of the Bhabha–Gupta equation (Bhabha 1952, Gupta 1954) with a proper choice of the free parameters present in the equation, but that this choice makes the total charge in the *free* theory indefinite. This observation prompted the following question in relation to the spin $\frac{3}{2}$ equation recently suggested by Fisk and Tait (1973): is the absence of acausality in the propagation of this field also achieved at the expense of having an indefinite total charge even in the free field case? The answer turned out to be in the affirmative. This paper presents our calculations leading to the above-stated results in both the Bhabha–Gupta and Fisk–Tait theories. These results are of particular interest in that they seem to suggest that for causal propagation of half-integer higher spin fields coupled minimally to external electromagnetic fields, an indefinite total charge for the free field is a prerequisite. This would in turn necessitate an indefinite metric in the quantization of the free field†. Furthermore, since the total charge is

† With the progress in the recent past in indefinite metric theories (Sudarshan 1968, 1972, Nakanishi 1972), one can no longer ignore them as unacceptable, since physically meaningful results can still be got out of them.

positive in half-integer spin theories with a single mass and spin, as noted long ago by Pauli (1940), the requirement of an indefinite total charge would also mean that more than one spin/mass be present in theories which remain causal in the presence of interaction. This is indeed the case with the two theories considered here: the Bhabha–Gupta field involves both spin $\frac{3}{2}$ and spin $\frac{1}{2}$ parts with two masses which are in general different, and the Fisk–Tait field has two distinct spin $\frac{3}{2}$ solutions.

The investigation of the Bhabha–Gupta field is presented in § 2. The condition for positivity of charge is first noted and it is then shown that this condition has to be violated if causality of propagation is to be ensured. The Fisk–Tait theory is considered in § 3.

2. The Bhabha-Gupta equation

The Lagrangian density for the free Bhabha–Gupta field having two different mass states and spins $\frac{3}{2}$ and $\frac{1}{2}$ is given by†

$$\mathscr{L} = -\bar{\psi}^{\mu}(\gamma \cdot p+m)\psi_{\mu}+\tfrac{1}{3}\bar{\psi}^{\mu}(p^{\nu}\gamma_{\mu}+\gamma^{\nu}p_{\mu})\psi_{\nu}-\tfrac{1}{3}\bar{\psi}^{\mu}\gamma_{\mu}(\gamma \cdot p-m)\gamma \cdot \psi-a\bar{\phi}(\gamma \cdot p+\lambda m)\phi$$
$$-d(\bar{\phi}p_{\mu}\psi^{\mu}+\bar{\psi}^{\mu}p_{\mu}\phi). \tag{1}$$

Here ψ^{μ} is a vector–spinor and ϕ is a Dirac spinor; $\bar{\psi}^{\mu} = \psi^{\dagger\mu}\gamma^{0}$ and $\bar{\phi} = \phi^{\dagger}\gamma^{0}$, where the dagger denotes the Hermitian conjugate. The constants a, λ and d are real and arbitrary. The expression (1) for $\mathscr{L}$ leads to the Euler–Lagrange equations

$$(\gamma \cdot p+m)\psi_{\mu}-\tfrac{1}{3}(\gamma^{\nu}p_{\mu}+p^{\nu}\gamma_{\mu})\psi_{\nu}+\tfrac{1}{3}\gamma_{\mu}(\gamma \cdot p-m)\gamma \cdot \psi+dp_{\mu}\phi = 0 \tag{2}$$

and

$$a(\gamma \cdot p+\lambda m)\phi+dp \cdot \psi = 0. \tag{3}$$

Multiplying (2) successively by $(\gamma \cdot p)\gamma^{\mu}$ and p^{μ} from the left and taking the difference of the resulting equations, we obtain

$$p \cdot \psi = 0, \tag{4a}$$

and by substituting (4*a*) in one of those resulting equations, we get

$$\gamma \cdot \psi = -3d\lambda\phi. \tag{4b}$$

When (4*a*) and (4*b*) are substituted in (2) and (3) the latter equations reduce respectively to

$$(\gamma \cdot p+m)\psi_{\mu}+d(1+\lambda)(p_{\mu}+\lambda m\gamma_{\mu})\phi = 0 \tag{4c}$$

and

$$(\gamma \cdot p+\lambda m)\phi = 0. \tag{4d}$$

The conserved charge-current vector j^{ρ} derived from (1) is

$$j^{\rho} = -\bar{\psi}^{\nu}\gamma^{\rho}\psi_{\nu}+\tfrac{1}{3}(\bar{\psi} \cdot \gamma)\psi^{\rho}+\tfrac{1}{3}\bar{\psi}^{\rho}(\gamma \cdot \psi)-\tfrac{1}{3}(\bar{\psi} \cdot \gamma)\gamma^{\rho}(\gamma \cdot \psi)-a\bar{\phi}\gamma^{\rho}\phi-d(\bar{\phi}\psi^{\rho}+\bar{\psi}^{\rho}\phi). \tag{5}$$

†Notation:

$$A \cdot B = A^{\mu}B_{\mu} = g^{\mu\nu}A_{\nu}B_{\mu} = A_0B_0-\boldsymbol{A} \cdot \boldsymbol{B};\ g^{00} = -g^{11} = -g^{22} = -g^{33} = 1\ .$$

The gamma matrices obey the relation $\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$. $\gamma^{5} = \gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}$ and $\sigma^{\mu\nu} = \frac{1}{2}\mathrm{i}(\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu})$. γ^{0} is Hermitian while γ^{k} is anti-Hermitian. $p_{\mu} = -\mathrm{i}(\partial/\partial x^{\mu})$.

To study the positivity or otherwise of the total charge, $\int j^0\, d^3x$, it is enough to consider the rest frame in which $\boldsymbol{p} = 0$. Then equation (4a) shows that

$$\psi^0 = 0. \tag{6}$$

Using (6) in (5) we get

$$\int j^0\, d^3x = \int d^3x[(\boldsymbol{\psi}^\dagger \,.\, \boldsymbol{\psi}) - \tfrac{1}{3}(\overline{\boldsymbol{\psi}} \,.\, \boldsymbol{\gamma})\gamma^0(\boldsymbol{\gamma} \,.\, \boldsymbol{\psi}) - a(\phi^\dagger\phi)]$$

$$= \int d^3x\{[\boldsymbol{\psi} + \tfrac{1}{3}\boldsymbol{\gamma}(\boldsymbol{\gamma} \,.\, \boldsymbol{\psi})]^\dagger \,.\, [\boldsymbol{\psi} + \tfrac{1}{3}\boldsymbol{\gamma}(\boldsymbol{\gamma} \,.\, \boldsymbol{\psi})] - a(\phi^\dagger\phi)\}. \tag{7}$$

This expression shows that the total charge is positive for $a < 0$, while it is indefinite for $a > 0$. One can easily convince oneself that this indefiniteness is not eliminated by the remaining equations in (4).

We now introduce minimal electromagnetic interaction through the usual prescription $p_\mu \to \pi_\mu = p_\mu - eA_\mu$, A_μ being the electromagnetic potential. Equations (2) and (3) then go over respectively into

$$(\gamma \,.\, \pi + m)\psi_\mu - \tfrac{1}{3}(\gamma^\nu\pi_\mu + \pi^\nu\gamma_\mu)\psi_\nu + \tfrac{1}{3}\gamma_\mu(\gamma \,.\, \pi - m)\gamma \,.\, \psi + d\pi_\mu\phi = 0 \tag{8}$$

and

$$a(\gamma \,.\, \pi + \lambda m)\phi + d\pi \,.\, \psi = 0. \tag{9}$$

The equations corresponding to (4a) and (4b) are respectively

$$\pi \,.\, \psi = m\alpha \tag{10a}$$

and

$$\gamma \,.\, \psi = [2 - (3d^2/a)]\alpha - 3d\lambda\phi. \tag{10b}$$

Here

$$\alpha = \frac{e}{m^2}\left(\frac{d}{2}(\sigma \,.\, F)\phi + \mathrm{i}(\gamma \,.\, F \,.\, \psi) - \tfrac{1}{6}(\sigma \,.\, F)(\gamma \,.\, \psi)\right), \tag{11}$$

$F_{\mu\nu}$ being the electromagnetic field tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Using the relation

$$\mathrm{i}\gamma^5\gamma \,.\, \hat{F} \,.\, \psi = \mathrm{i}\gamma \,.\, F \,.\, \psi - \tfrac{1}{2}(\sigma \,.\, F)(\gamma \,.\, \psi)$$

where $\hat{F}^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}$ is the tensor dual to $F^{\mu\nu}$, we may rewrite (11) in the form

$$\alpha = \frac{2}{3}\frac{ie}{m^2}\gamma \,.\, F \,.\, \psi + \frac{1}{3}\frac{ie}{m^2}\gamma^5\gamma \,.\, \hat{F} \,.\, \psi + \frac{ed}{2m^2}\sigma \,.\, F\phi. \tag{11a}$$

We employ the shock-wave formalism of Madore and Tait (1973) to test whether the above equations suffer from acausality of propagation or not. This method—an alternative but equivalent one to that of Velo and Zwanziger (1969a)—exploits the fact that the characteristic surfaces are surfaces across which there can exist discontinuities in the highest order derivatives appearing in the wave equation (Courant and Hilbert 1962). In the context of equations (8) and (9), this means that while ψ_μ and ϕ are continuous across a characteristic surface σ, $\partial_\mu\psi_\nu$ and $\partial_\mu\phi$ may be discontinuous. Using square brackets to denote the magnitudes of the discontinuities across σ, we have†

$$[\partial_\mu \psi_\nu] = \xi_\mu k_\nu \qquad \text{and} \qquad [\partial_\mu \phi] = \xi_\mu K,$$

while $[\psi_\mu] = [\phi] = 0$. The ξ_μ here are the components of the normal to any characteristic surface and k_ν denotes a vector–spinor (like ψ_ν).

Now taking the discontinuities across σ of equations (8), (9) and (10*a*) we get, respectively,

$$(\gamma . \xi)k_\mu - \tfrac{1}{3}\xi_\mu(\gamma . k) - \tfrac{1}{3}\gamma_\mu(\xi . k) + \tfrac{1}{3}\gamma_\mu(\gamma . \xi)(\gamma . k) + d\xi_\mu K = 0, \tag{12a}$$

$$a(\gamma . \xi)K + d(\xi . k) = 0 \tag{12b}$$

and

$$\xi . k = 0. \tag{12c}$$

(In writing the last equation we have used the fact that α of equation (11*a*) is free of derivatives.) Further, differentiating both sides of (10*b*) and then taking discontinuities, we get

$$\xi_\mu(\gamma . k) = \xi_\mu\{[2-(3d^2/a)]\tilde{\alpha} - 3d\lambda K\}, \tag{12d}$$

where $\tilde{\alpha} = \alpha(\psi_\mu \to k_\mu, \phi \to K)$. From (12*b*) and (12*c*) it follows that $(\gamma . \xi)K = 0$ and hence

$$\xi^2 K = 0. \tag{13}$$

Now, we are interested in the conditions for non-existence of spacelike characteristic surfaces. If such a surface did exist and ξ_μ were the normal to it ($\xi^2 > 0$), then one would have from (13) that

$$K = 0. \tag{14}$$

Further, with $\xi^2 \neq 0$, equation (12*d*) would give

$$\gamma . k = [2-(3d^2/a)]\tilde{\alpha} - 3d\lambda K. \tag{15}$$

Substituting (12*c*) and (14) in (12*a*) we obtain

$$(\gamma . \xi)k_\mu - \tfrac{1}{3}\xi_\mu(\gamma . k) + \tfrac{1}{3}\gamma_\mu(\gamma . \xi)(\gamma . k) = 0.$$

This equation enables one to factor out the μ-dependence of k_μ and write it as

$$k_\mu = [\xi_\mu - \gamma_\mu(\gamma . \xi)]\chi, \qquad \chi = -\frac{1}{3}\frac{(\gamma . \xi)(\gamma . k)}{\xi^2}. \tag{16}$$

Note that χ is a spinor. Feeding the above expression for k_μ into (15) and using $K = 0$ from equation (14), we obtain

$$\xi^2\left\{\xi^2 + \left[\frac{2}{3}em^{-2}\left(1-\frac{3d^2}{2a}\right)\right]^2(\hat{F} . \xi)^2\right\}\chi = 0. \tag{17}$$

It is easy to verify that for any $a < 0$ this equation admits solutions with $(\xi^0)^2 > \boldsymbol{\xi}^2$ and hence leads to acausal propagation. On the other hand, if $a > 0$ we can choose the arbitrary parameter d such that $d^2 = \frac{2}{3}a$. Then

$$(\xi^2)^2\chi = 0 \Rightarrow \chi = 0 \Rightarrow k_\mu = 0; \tag{18}$$

† For notation and other details see Madore and Tait (1973).

so that the discontinuities k_μ and K both vanish for any $\xi^2 \neq 0$, ie there can be no characteristic surfaces with $\xi^2 \neq 0$. We see thus that if the charge in the free field theory is required to be positive definite ($a < 0$) one cannot have causality in the presence of interaction, while by giving up definiteness of charge ($a > 0$) and making the choice $d^2 = \frac{2}{3}a$, causality can be achieved.

3. The Fisk–Tait equation

Recently Fisk and Tait (1973) have proposed an equation for spin $\frac{3}{2}$ particles which has been shown to remain causal with minimal electromagnetic coupling. We now demonstrate that in their theory the total charge is indefinite.

The wavefunction employed is a 24-component antisymmetric tensor–spinor $\psi_\alpha^{\mu\nu} = -\psi_\alpha^{\nu\mu}$, obeying the equation

$$-\tfrac{4}{3}\gamma\cdot p\psi^{\mu\nu} - \tfrac{1}{3}(\gamma\cdot p)(\gamma^\mu\gamma_\rho g^\nu{}_\sigma - \gamma^\nu\gamma_\rho g^\mu{}_\sigma)\psi^{\sigma\rho} + \tfrac{1}{3}(\gamma^\mu p_\sigma g^\nu{}_\rho - \gamma^\nu p_\sigma g^\mu{}_\rho - p^\mu\gamma_\sigma g^\nu{}_\rho + p^\nu\gamma_\sigma g^\mu{}_\rho)\psi^{\sigma\rho} + m\psi^{\mu\nu} = 0. \tag{19}$$

The subsidiary conditions which follow from (19) are

$$\gamma_\mu\gamma_\nu\psi^{\mu\nu} = 0, \tag{20a}$$

$$p_\mu\gamma_\nu\psi^{\mu\nu} = 0. \tag{20b}$$

It is easily verified using (19) that the current

$$j^\rho = -\tfrac{4}{3}\bar{\psi}^{\mu\nu}\gamma^\rho\psi_{\mu\nu} - \tfrac{2}{3}\bar{\psi}^{\mu\nu}\gamma^\rho\gamma_\mu\gamma^\sigma\psi_{\nu\sigma} + \tfrac{2}{3}\bar{\psi}^{\mu\nu}\gamma_\mu\psi^\rho{}_\nu - \tfrac{2}{3}\bar{\psi}^{\rho\nu}\gamma^\sigma\psi_{\sigma\nu}, \qquad (\bar{\psi}^{\mu\nu} = \psi^{\dagger\mu\nu}\gamma^0) \tag{21}$$

is Hermitian and conserved.

To demonstrate that the total charge is indefinite when $\psi^{\mu\nu}$ satisfies the equation of motion, it is enough to consider the rest frame in which $\boldsymbol{p} = 0$. Then (20b) and (20a) reduce to

$$\gamma_i\psi^{0i} = 0, \qquad \gamma_i\gamma_j\psi^{ij} = 0, \tag{22}$$

and when this is used the total charge can be obtained as

$$\int j^0\,\mathrm{d}^3x = \int \mathrm{d}^3x[2\psi_{0j}^\dagger\psi_{0j} - \tfrac{4}{3}\psi_{ij}^\dagger\psi_{ij} + \tfrac{2}{3}(\gamma_i\psi_{ij})^\dagger(\gamma_k\psi_{kj})]$$

$$= 2\int \mathrm{d}^3x[(\psi_{01}^\dagger\psi_{01} + \psi_{02}^\dagger\psi_{02} + \psi_{03}^\dagger\psi_{03}) - (\psi_{12}^\dagger\psi_{12} + \psi_{23}^\dagger\psi_{23} + \psi_{31}^\dagger\psi_{31})].$$

This expression is clearly indefinite and it is easily seen that equations (22) do not eliminate the indefiniteness.

4. Discussion

The above results lead one to speculate that retention of causality in the presence of electromagnetic interaction may in general be impossible (for half-integer spins $>\frac{1}{2}$) unless one starts with a free field theory which is reducible (in the sense of containing more than one spin value) and wherein the total charge has no definite sign. (In both theories considered here the contributions of the two spins to the total charge are found to be of opposite sign.) There is another matter which remains to be considered apart from the above general question. It is known from the example of the vector field with minimal plus anomalous magnetic moment coupling to the electromagnetic field that even when the field equations do not lead to any space-like characteristic surfaces (Velo and Zwanziger 1969b) the field may have tachyonic modes (Mathews 1974, Mathews and Seetharaman 1973, Tsai and Yildiz 1971). Does such a phenomenon arise in the theories here considered? Investigations on this point are in progress and the results will be reported separately.

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